

The contact process with semi-infected state on the complete graph

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Abstract: In this paper we are concerned with the contact process with semi-infected state on the complete graph C_n with n vertices. In our model, each vertex is in one of three states that ‘healthy’, ‘semi-infected’ or ‘wholly-infected’. Only wholly-infected vertices can infect others. A healthy vertex becomes semi-infected when being infected while a semi-infected vertex becomes wholly-infected when being further infected. Each (semi- and wholly-) infected vertex becomes healthy at constant rate. Our main result shows the phase transition for the time wholly-infected vertices wait for to die out. Conditioned on all the vertices are wholly-infected when $t = 0$, we show that wholly-infected vertices survive for $\exp\{O(n)\}$ units of time when the infection rate $\lambda > 4$ while die out in $O(\log n)$ units of time when $\lambda < 4$.

Keywords: Contact process, semi-infected, complete graph, phase transition.

1 Introduction

In this paper we are concerned with the contact process with semi-infected state on the complete graph. A complete graph is a finite graph such that for any two vertices there is an edge connecting them. For later use, for integer $n \geq 1$, we denote by C_n the complete graph with n vertices and denote by $\{1, 2, \dots, n\}$ the vertices set of C_n .

The contact process with semi-infected state on C_n is a continuous-time Markov process with state space $\{0, 1, 2\}^{C_n}$, i. e. at each vertex there is a spin taking value from $\{0, 1, 2\}$. For any configuration $\eta \in \{0, 1, 2\}^{C_n}$ and $1 \leq i \leq n$, we denote by $\eta(i)$ the value of the spin at the vertex i . For any $t \geq 0$, we denote by η_t the configuration of our process at moment t . For $\eta \in \{0, 1, 2\}^{C_n}$, $1 \leq i \leq n$ and $0 \leq l \leq 2$, we define $\eta^{i,l} \in \{0, 1, 2\}^{C_n}$ as follows.

$$\eta^{i,l}(j) = \begin{cases} \eta(j), & \text{if } j \neq i, \\ l, & \text{if } j = i. \end{cases}$$

The generator Ω of $\{\eta_t\}_{t \geq 0}$ has the form

$$\Omega f(\eta) = \sum_{1 \leq i \leq n} \sum_{l=0,1,2} H(\eta, i, l) [f(\eta^{i,l}) - f(\eta)] \quad (1.1)$$

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for any continuous function f on $\{0, 1, 2\}^{C_n}$. That is to say, at each moment that the configuration of the process jumps, only one spin changes value. Conditioned on the current configuration η , the process jumps to $\eta^{i,l}$ at rate $H(\eta, i, l)$. $H(\eta, i, l)$ is defined as

$$H(\eta, i, l) = \begin{cases} 1 & \text{if } l = 0 \text{ and } \eta(i) \neq 0, \\ \frac{\lambda}{n} |\{j : \eta(j) = 2\}| & \text{if } l = 2 \text{ and } \eta(i) = 1, \\ \frac{\lambda}{n} |\{j : \eta(j) = 2\}| & \text{if } l = 1 \text{ and } \eta(i) = 0, \\ 0 & \text{else,} \end{cases}$$

where λ is a positive constant called the infection rate and $|A|$ is the cardinality of the set A .

Intuitively, the process describes the spread of an epidemic, where each individual is in one of the three states that healthy, semi-infected or wholly-infected. For details, vertices in state 0 are healthy while vertices in state 1 are semi-infected and vertices in state 2 are wholly-infected. Each infected vertex, no matter semi- or wholly-, waits for an exponential time with rate 1 to become healthy. Wholly-infected vertices have the ability to infect others. A healthy vertex is infected at rate proportional to the number of wholly-infected vertices and becomes a semi-infected vertex when being infected. A semi-infected vertex is further infected at rate proportional to the number of wholly-infected vertices and becomes a wholly-infected vertex when being infected.

According to the spatial homogeneity of the process $\{\eta_t\}_{t \geq 0}$, we only care about the numbers of vertices in state 1 and 2. Hence we define

$$B_t = |\{i : \eta_t(i) = 2\}| \text{ and } G_t = |\{i : \eta_t(i) = 1\}|$$

for any $t \geq 0$. According to the generator of $\{\eta_t\}_{t \geq 0}$ given in Equation (1.1) and the definition of $H(\eta, i, l)$, the transition rates function of $\{(B_t, G_t)\}_{t \geq 0}$ is given by

$$(B, G) \text{ jumps to } \begin{cases} (B-1, G) & \text{at rate } B, \\ (B, G-1) & \text{at rate } G, \\ (B+1, G-1) & \text{at rate } \frac{\lambda}{n} BG, \\ (B, G+1) & \text{at rate } \frac{\lambda}{n} B(n-B-G). \end{cases}$$

The process $\{\eta_t\}_{t \geq 0}$ is an extension of the classic contact process introduced in [6] by Harris, where vertices are distinguished as healthy ones and infected ones such that a healthy one is infected at rate proportional to the number of infected neighbors while an infected one becomes healthy at rate one. For a detailed survey of the study of the classic contact process, see Chapter 6 of [7] and Part one of [8].

In [9], Peterson studies the contact process on the complete graph with vertex-dependent infection rates, containing the classic contact process as a case. It is shown in [9] that there is a critical value λ_c of the infection rate λ such that when $\lambda < \lambda_c$ the process dies out before $O(\log n)$ units of time with high probability while when $\lambda > \lambda_c$ the process survives for $\exp\{O(n)\}$ units of time with high probability. We are inspired by [9] a lot. Our main result shows that similar phase transition with that in [9] occurs for the contact process with semi-infected state. The precise value of the critical infection rate is also given. For mathematical details, see Section 2.

2 Main result

In this section we give the main result of this paper. We care about the first moment that there is no vertex in state 2, hence we define

$$\tau = \inf\{t : B_t = 0\}.$$

Since vertices in state 1 can not infect others, after the moment τ , the epidemic dies out in $\log n$ units of time with high probability, which depends on the fact that the maximum of n independent exponential times with rate one is with order $\log n$ with high probability. As a result, whether the epidemic survives for a long time, for instance $\exp\{O(n)\}$ units of time, depends on τ .

For each $n \geq 1$ and any $\lambda > 0$, we denote by $P_{\lambda,n}$ the contact process with semi-infected state on C_n with infection rate λ . We write $P_{\lambda,n}$ as $P_{\lambda,n}^{B,G}$ when $(B_0, G_0) = (B, G)$. For given $b, g \in [0, 1]$, we write $(B, G) = (nb, ng)$ when $(B, G) = (\lfloor nb \rfloor, \lfloor ng \rfloor)$ for simplicity. We have the following main result which shows phase transition for τ .

Theorem 2.1. *When $\lambda > 4$, there exists a constant $C = C(\lambda) > 0$ such that*

$$\lim_{n \rightarrow +\infty} P_{\lambda,n}^{n,0}(\tau > e^{Cn}) = 1 \quad (2.1)$$

while when $\lambda < 4$,

$$\lim_{n \rightarrow +\infty} P_{\lambda,n}^{n,0}(\tau < (1 + \theta) \log n) = 1. \quad (2.2)$$

for any $\theta > 0$.

According to Theorem 2.1, conditioned on all the vertices are wholly-infected at $t = 0$, the wholly-infected vertices survive for $\exp\{O(n)\}$ units of time when $\lambda > 4$ while die out in $O(\log n)$ units of time when $\lambda < 4$.

We are inspired by former references about the contact process on finite sets to prove Theorem 2.1. According to the main results in [9], the classic contact process on the complete graph C_n survives for $\exp\{O(n)\}$ units of time when $\lambda > 1$ while dies out in $O(\log n)$ units of time when $\lambda < 1$. Foxall, Edwards and van den Driessche introduce the contact process on the complete graph incorporating monogamous dynamic partnerships in [5], where similar phase transition with that in [9] is shown and the precise value of the critical infection rate is given. Durrett and coworkers study the contact process on the finite lattice $[-N, N]^d$ in [1, 2, 3]. One of their main results is that the process survives for $\exp\{O(N^d)\}$ units of time when $\lambda > \lambda_c(d)$ while dies out in $O(\log N)$ units of time when $\lambda < \lambda_c(d)$, where $\lambda_c(d)$ is the minimum of the infection rates of the contact process on \mathbb{Z}^d with which the process survives forever with positive probability.

The proof of Theorem 2.1 is divided into three sections. In Section 3, we introduce an two-dimensional ODE

$$\begin{cases} \frac{d}{dt} b_t = F_1(b_t, g_t) \\ \frac{d}{dt} g_t = F_2(b_t, g_t) \end{cases}$$

and show that the solution (b_t, g_t) to this ODE is the mean field limit of $(\frac{B_t}{n}, \frac{G_t}{n})$ for t in any compact area as n grows to infinity. In Section 4, we give the proof of Equation (2.1). The proof relies heavily on the fact that when $\lambda > 4$ there exists (b, g) such that $F_1(b, g) > 0$

and $F_2(b, g) > 0$. In Section 5, we give the proof of Equation (2.2). The proof relies heavily on the fact that $(0, 0)$ is the unique equilibrium state of the ODE when $\lambda < 4$.

For later use, at the end of this section we give a lemma which shows that the contact process with semi-infected state is monotonic under a specific partial order on \mathbb{Z}^2 . For $(B_1, G_1), (B_2, G_2) \in \mathbb{Z}^2$, we write $(B_1, G_1) \succeq (B_2, G_2)$ when and only when $B_1 \geq B_2$ and $B_1 + G_1 \geq B_2 + G_2$. It is easy to check that \succeq is a partial order on \mathbb{Z}^2 . We write $(B_2, G_2) \preceq (B_1, G_1)$ when $(B_1, G_1) \succeq (B_2, G_2)$. We have the following lemma.

Lemma 2.2. *For $j=1, 2$, if $\{(B_t^j, G_t^j)\}_{t \geq 0}$ is the contact process with semi-infected state with initial state (B_0^j, G_0^j) and $(B_0^1, G_0^1) \succeq (B_0^2, G_0^2)$, then $(B_t^1, G_t^1) \succeq (B_t^2, G_t^2)$ for any $t \geq 0$ in the sense of coupling.*

Proof. Conditioned on $(B^1, G^1) \succeq (B^2, G^2)$, we couple the two processes as follows.

$$\begin{aligned} & \left((B^1, G^1), (B^2, G^2) \right) \rightarrow \\ & \left\{ \begin{array}{ll} \left((B^1 - 1, G^1), (B^2 - 1, G^2) \right) & \text{at rate } B^2, \\ \left((B^1 - 1, G^1), (B^2, G^2) \right) & \text{at rate } B^1 - B^2, \\ \left((B^1, G^1 - 1), (B^2, G^2 - 1) \right) & \text{at rate } \min\{G^1, G^2\}, \\ \left((B^1, G^1 - 1), (B^2, G^2) \right) & \text{at rate } G^1 - \min\{G^1, G^2\}, \\ \left((B^1, G^1), (B^2, G^2 - 1) \right) & \text{at rate } G^2 - \min\{G^1, G^2\}, \\ \left((B^1 + 1, G^1 - 1), (B^2 + 1, G^2 - 1) \right) & \text{at rate } \frac{\lambda}{n} \min\{B^1 G^1, B^2 G^2\}, \\ \left((B^1 + 1, G^1 - 1), (B^2, G^2) \right) & \text{at rate } \frac{\lambda}{n} (B^1 G^1 - \min\{B^1 G^1, B^2 G^2\}), \\ \left((B^1, G^1), (B^2 + 1, G^2 - 1) \right) & \text{at rate } \frac{\lambda}{n} (B^2 G^2 - \min\{B^1 G^1, B^2 G^2\}), \\ \left((B^1, G^1 + 1), (B^2, G^2 + 1) \right) & \text{at rate } a, \\ \left((B^1, G^1 + 1), (B^2, G^2) \right) & \text{at rate } \frac{\lambda}{n} B^1 (n - B^1 - G^1) - a, \\ \left((B^1, G^1), (B^2, G^2 + 1) \right) & \text{at rate } \frac{\lambda}{n} B^2 (n - B^2 - G^2) - a, \end{array} \right. \end{aligned}$$

where

$$a = \frac{\lambda}{n} \min\{B^1(n - B^1 - G^1), B^2(n - B^2 - G^2)\}.$$

By direct calculation, it is easy to check that the Markov process with the above transition rates function is a coupling of (B_t^1, G_t^1) and (B_t^2, G_t^2) while all the state transitions hold the property that $(B^1, G^1) \succeq (B^2, G^2)$. □

3 Mean field limit

In this section we introduce an ODE, the solution to which is the mean field limit of $(\frac{B_t}{n}, \frac{G_t}{n})$ as n grows to infinity. For later use, for any $x = (b, g) \in \mathbb{R}^2$, we use $\|x\|_1$ to denote $|b| + |g|$, which is the l_1 norm of $x = (b, g)$.

We consider the following two-dimensional ODE with initial condition $(b_0, g_0) \in [0, 1] \times [0, 1]$.

$$\begin{cases} \frac{d}{dt}b_t = -b_t + \lambda b_t g_t, \\ \frac{d}{dt}g_t = -g_t - \lambda b_t g_t + \lambda b_t(1 - b_t - g_t), \end{cases}$$

where $\lambda > 0$. For simplicity, we write the above ODE as

$$\frac{d}{dt}(b_t, g_t) = F(b_t, g_t), \quad (3.1)$$

where $F = (F_1, F_2)$ such that $F_1(b, g) = -b + \lambda b g$ and $F_2(b, g) = -g - \lambda b g + \lambda b(1 - b - g)$.

Later we will show that $\frac{1}{n}(B_t, G_t)$ converges to the solution (b_t, g_t) to ODE (3.1) in probability. By direct calculation, it is easy to check that

$$F(b, g) = (0, 0)$$

has the unique solution $(0, 0)$ when $\lambda < 4$ while has two solutions in $(0, 1) \times (0, 1)$ when $\lambda > 4$, which intuitively explains why the critical value of our process is 4.

We define $\Lambda = \{(b, g) : b \geq 0, g \geq 0, b + g \leq 1\} \subseteq \mathbb{R}^2$, then we have the following lemma.

Lemma 3.1. *The solution (b_t, g_t) to ODE (3.1) with initial condition $(b_0, g_0) \in \Lambda$ exists for $t \in [0, +\infty)$ and is unique. Furthermore, for any $t \geq 0$, $(b_t, g_t) \in \Lambda$.*

Proof. It is easy to check that F satisfies the local Lipschitz condition under the norm $\|\cdot\|_1$, according to which the uniqueness of the solution holds. On the boundary of Λ , it is easy to check that any vector of the vector-field of ODE (3.1) points to the inner of Λ , hence the solution is absorbed in the area Λ . On the area Λ , it is easy to check that F satisfies the global Lipschitz condition, hence the solution exists for $t \in [0, +\infty)$. \square

The next lemma shows that $(\frac{B_t}{n}, \frac{G_t}{n})$ converges to the solution (b_t, g_t) to the ODE (3.1) in probability as n grows to infinity.

Lemma 3.2. *Let $\{(b_t, g_t)\}_{t \geq 0}$ be the solution to the ODE (3.1) with initial condition $(b_0, g_0) \in \Lambda$, then for any $T > 0$ and $\epsilon > 0$, there exist constants $C_1 = C_1(T, \epsilon)$ and $N_1 = N_1(T, \epsilon)$ such that*

$$P_{\lambda, n}^{nb_0, ng_0} \left(\sup_{0 \leq t \leq T} \left\| \left(\frac{B_t}{n}, \frac{G_t}{n} \right) - (b_t, g_t) \right\|_1 > \epsilon \right) \leq \frac{C_1}{n}$$

for any $n \geq N_1$.

Note that C_1 and N_1 do not depend on the choice of (b_0, g_0) .

The proof of Lemma 3.2 follows the analysis introduced by Ethier and Kurtz to construct the theory of density-dependent population model (See Chapter 11 of [4]). Readers familiar with this theory can skip the following proof.

Proof of Lemma 3.2. By the definition of (b_t, g_t) ,

$$\begin{cases} b_t = b_0 + \int_0^t F_1(b_s, g_s) ds, \\ g_t = g_0 + \int_0^t F_2(b_s, g_s) ds. \end{cases} \quad (3.2)$$

Let $\{N_j(t) : t \geq 0\}_{j=1,2,3,4}$ be four independent copies of the Poisson process with rate one, then according to the transition rates function of $\{B_t, G_t\}_{t \geq 0}$ and Theorem 6.4.1 of [4], we can write (B_t, G_t) as

$$\begin{cases} B_t = \lfloor nb_0 \rfloor - N_1(\int_0^t B_s ds) + N_3(\int_0^t \frac{\lambda}{n} B_s G_s ds), \\ G_t = \lfloor ng_0 \rfloor - N_2(\int_0^t G_s ds) - N_3(\int_0^t \frac{\lambda}{n} B_s G_s ds) + N_4(\int_0^t \frac{\lambda}{n} B_s (n - B_s - G_s) ds). \end{cases} \quad (3.3)$$

For $j = 1, 2, 3, 4$, we define $\tilde{N}_j(t) = N_j(t) - t$, then $\{\tilde{N}_j(t) : t \geq 0\}$ is a martingale with $E\tilde{N}_j(t) = 0$ and $E[\tilde{N}_j^2(t)] = t$. According to Equations (3.2), (3.3) and the definition of $F = (F_1, F_2)$,

$$\begin{cases} \frac{B_t}{n} - b_t = \frac{\lfloor nb_0 \rfloor - nb_0}{n} + \int_0^t F_1(\frac{B_s}{n}, \frac{G_s}{n}) - F_1(b_s, g_s) ds - \frac{\tilde{N}_1(\int_0^t B_s ds)}{n} + \frac{\tilde{N}_3(\int_0^t \frac{\lambda}{n} B_s G_s ds)}{n}, \\ \frac{G_t}{n} - g_t = \frac{\lfloor ng_0 \rfloor - ng_0}{n} + \int_0^t F_2(\frac{B_s}{n}, \frac{G_s}{n}) - F_2(b_s, g_s) ds \\ - \frac{\tilde{N}_2(\int_0^t G_s ds)}{n} - \frac{\tilde{N}_3(\int_0^t \frac{\lambda}{n} B_s G_s ds)}{n} + \frac{\tilde{N}_4(\int_0^t \frac{\lambda}{n} B_s (n - B_s - G_s) ds)}{n}. \end{cases} \quad (3.4)$$

Since $B_t \leq n$, $|\tilde{N}_1(\int_0^t B_s ds)| \leq \max_{0 \leq s \leq nt} |\tilde{N}_1(s)|$. For similar reasons,

$$\begin{aligned} |\tilde{N}_2(\int_0^t B_s ds)| &\leq \max_{0 \leq s \leq nt} |\tilde{N}_1(s)|, \\ |\tilde{N}_3(\int_0^t \frac{\lambda}{n} B_s G_s ds)| &\leq \max_{0 \leq s \leq \lambda nt} |\tilde{N}_3(s)|, \\ |\tilde{N}_4(\int_0^t \frac{\lambda}{n} B_s G_s ds)| &\leq \max_{0 \leq s \leq \lambda nt} |\tilde{N}_4(s)|. \end{aligned}$$

It is easy to check that $F = (F_1, F_2)$ satisfies the global Lipschitz condition on Λ under the norm $\|\cdot\|_1$, hence there exists $K > 0$ such that

$$\|F(b_1, g_1) - F(b_2, g_2)\|_1 \leq K\|(b_1, g_1) - (b_2, g_2)\|_1$$

for any $(b_1, g_1), (b_2, g_2) \in \Lambda$. It is obviously that $\frac{1}{n}(B_t, G_t) \in \Lambda$ for any $t \geq 0$. As we have shown in Lemma 3.1, $(b_t, g_t) \in \Lambda$ for any $t \geq 0$. As a result, by Equation (3.4),

$$\|(\frac{B_t}{n}, \frac{G_t}{n}) - (b_t, g_t)\|_1 \leq \int_0^t K\|(\frac{B_s}{n}, \frac{G_s}{n}) - (b_s, g_s)\|_1 ds + \frac{M(\lambda, n, t)}{n} \quad (3.5)$$

for any $t \geq 0$, where

$$M(\lambda, n, t) = 2 + \max_{0 \leq s \leq nt} |\tilde{N}_1(s)| + \max_{0 \leq s \leq nt} |\tilde{N}_2(s)| + 2 \max_{0 \leq s \leq \lambda nt} |\tilde{N}_3(s)| + \max_{0 \leq s \leq \lambda nt} |\tilde{N}_4(s)|.$$

Since $M(\lambda, n, t)$ increases with t ,

$$\|(\frac{B_t}{n}, \frac{G_t}{n}) - (b_t, g_t)\|_1 \leq \int_0^t K\|(\frac{B_s}{n}, \frac{G_s}{n}) - (b_s, g_s)\|_1 ds + \frac{M(\lambda, n, T)}{n}$$

for $0 \leq t \leq T$. Then by Grownwall's inequality,

$$\|(\frac{B_t}{n}, \frac{G_t}{n}) - (b_t, g_t)\|_1 \leq \frac{M(\lambda, n, T)}{n} e^{Kt} \quad (3.6)$$

for $0 \leq t \leq T$. By Equation (3.6),

$$\sup_{0 \leq t \leq T} \left\| \left(\frac{B_t}{n}, \frac{G_t}{n} \right) - (b_t, g_t) \right\|_1 \leq \frac{M(\lambda, n, T)}{n} e^{KT}. \quad (3.7)$$

Since $\{\tilde{N}_1(t) : t \geq 0\}$ is a martingale, according to Doob's inequality,

$$P\left(\max_{0 \leq s \leq nT} |\tilde{N}_1(s)| \geq n\epsilon_0\right) \leq \frac{1}{n^2 \epsilon_0^2} E[\tilde{N}_1^2(nT)] = \frac{T}{n\epsilon_0^2}$$

for any $\epsilon_0 > 0$. For given $\epsilon > 0$, we choose $\epsilon_0 = \frac{\epsilon}{8e^{KT}}$, then

$$P\left(\max_{0 \leq s \leq nT} |\tilde{N}_1(s)| \geq \frac{n\epsilon}{8e^{KT}}\right) \leq \frac{64e^{2KT}T}{n\epsilon^2}.$$

According similar analysis,

$$\begin{aligned} P\left(\max_{0 \leq s \leq nT} |\tilde{N}_2(s)| \geq \frac{n\epsilon}{8e^{KT}}\right) &\leq \frac{64e^{2KT}T}{n\epsilon^2}, \\ P\left(2 \max_{0 \leq s \leq \lambda nT} |\tilde{N}_3(s)| \geq \frac{n\epsilon}{8e^{KT}}\right) &\leq \frac{256e^{2KT}\lambda T}{n\epsilon^2}, \\ P\left(\max_{0 \leq s \leq \lambda nT} |\tilde{N}_4(s)| \geq \frac{n\epsilon}{8e^{KT}}\right) &\leq \frac{64e^{2KT}\lambda T}{n\epsilon^2}. \end{aligned}$$

Therefore,

$$P\left(M(\lambda, n, T) \geq 2 + \frac{n\epsilon}{2e^{KT}}\right) \leq \frac{128e^{2KT}T + 320e^{2KT}\lambda T}{\epsilon^2}.$$

As a result,

$$P\left(\frac{M(\lambda, n, T)}{n} \geq \frac{\epsilon}{e^{KT}}\right) \leq \frac{C_1(T, \epsilon)}{n}$$

for $n \geq N_1(T, \epsilon) = \frac{4e^{KT}}{\epsilon}$, where

$$C_1(T, \epsilon) = \frac{128e^{2KT}T + 320e^{2KT}\lambda T}{\epsilon^2}.$$

Then by Equation (3.7),

$$P_{\lambda, n}^{n, 0}\left(\sup_{0 \leq t \leq T} \left\| \left(\frac{B_t}{n}, \frac{G_t}{n} \right) - (b_t, g_t) \right\|_1 > \epsilon\right) \leq P\left(\frac{M(\lambda, n, T)}{n} \geq \frac{\epsilon}{e^{KT}}\right) \leq \frac{C_1(T, \epsilon)}{n}$$

for $n \geq N_1(T, \epsilon)$ and the proof is complete. \square

The next lemma is crucial for the proof of Equation (2.2).

Lemma 3.3. *Let $\{(b_t, g_t)\}_{t \geq 0}$ be the solution to ODE (3.1) with initial condition $(b_0, g_0) = (1, 0)$, then when $\lambda < 4$,*

$$\lim_{t \rightarrow +\infty} \|(b_t, g_t)\|_1 = 0.$$

Proof. It is easy to check that $F_2(b, g) \geq 0$ when and only when $g \leq \frac{\lambda b(1-b)}{2\lambda b+1}$. Let $g^* = \max\{\frac{\lambda b(1-b)}{2\lambda b+1} : 0 \leq b \leq 1\}$, then it is easy to check that $g^* < \frac{1}{\lambda}$ when $\lambda < 4$. Let $\tilde{g} = \frac{1+g^*}{2}$,

then $g_0 = 0 < \tilde{g}$. For any $0 \leq b \leq 1$, $F_2(b, \tilde{g}) < 0$ since $\tilde{g} > g^*$. As a result, g_t can never exceed \tilde{g} since $\left. \frac{d}{dt}g_t \right|_{g_t=\tilde{g}} = F_2(b_t, \tilde{g}) < 0$. Therefore, $g_t \leq \tilde{g}$ for any $t \geq 0$. Then,

$$\frac{d}{dt}b_t \leq -b_t + \lambda\tilde{g}b_t = (\lambda\tilde{g} - 1)b_t$$

and hence

$$b_t \leq e^{(\lambda\tilde{g}-1)t} \quad (3.8)$$

for any $t \geq 0$. By Equation (3.8),

$$\frac{d}{dt}g_t = F_2(b_t, g_t) \leq -g_t + \lambda e^{(\lambda\tilde{g}-1)t}$$

for any $t \geq 0$. As a result,

$$\frac{d}{dt}(e^t g_t) \leq \lambda e^{\lambda\tilde{g}t}$$

and hence

$$g_t \leq \frac{e^{(\lambda\tilde{g}-1)t} - e^{-t}}{\tilde{g}}. \quad (3.9)$$

Since $\tilde{g} < \frac{1}{\lambda}$, $\lambda\tilde{g} - 1 < 0$. As a result, Lemma 3.3 follows from Equations (3.8) and (3.9) directly. \square

4 Proof of Equation (2.1)

In this section we give the proof of Equation (2.1). The intuitive idea of the proof is as follows. When $\lambda > 4$, it is easy to check that there exists (b_0, g_0) in the inner of Λ such that $F_1(b_0, g_0) > 0$ and $F_2(b_0, g_0) > 0$. Then, by analyzing the vector-field of ODE (3.1), it is easy to check that the solution $\{(b_t, g_t)\}_{t \geq 0}$ with initial condition (b_0, g_0) is absorbed in the area

$$\Lambda_1 = \{(b, g) : b \geq b_0, b + g \geq b_0 + g_0\}.$$

As shown in Lemma 3.2, conditioned on $(B_0, G_0) = (nb_0, ng_0)$, $\frac{1}{n}(B_t, G_t)$ is approximate to (b_t, g_t) . Hence (B_t, G_t) should stay in the area $n\Lambda_1$ for a long time. Since $(nb_0, ng_0) \preceq (n, 0)$ and $\{(B_t, G_t)\}_{t \geq 0}$ is monotone under the partial order \preceq by Lemma 2.2, (B_t, G_t) with $(B_0, G_0) = (n, 0)$ should stay in $n\Lambda$ for a longer time.

Our proof is the effort to make the above intuitive idea rigorous and show that the precise meaning of ‘long time’ is $\exp\{O(n)\}$ units of time. First we show the existence of (b_0, g_0) . We define

$$\tilde{\Lambda} = \{(b, g) : b > 0, g > 0, b + g < 1\}$$

as the inner of Λ .

Lemma 4.1. *When $\lambda > 4$, there exists $(b_0, g_0) \in \tilde{\Lambda}$ such that*

$$F_1(b_0, g_0) > 0 \text{ and } F_2(b_0, g_0) > 0,$$

where $F = (F_1, F_2)$ is as defined in ODE (3.1).

Proof. By direct calculation, when $\lambda > 4$, $(\frac{\lambda-2}{2\lambda}, \frac{1}{\lambda}) \in \tilde{\Lambda}$ and

$$F_2(\frac{\lambda-2}{2\lambda}, \frac{1}{\lambda}) > 0.$$

Then, we can choose sufficiently small positive β such that $F_2(\frac{\lambda-2}{2\lambda}, \frac{1}{\lambda} + \beta) > 0$ and $(\frac{\lambda-2}{2\lambda}, \frac{1}{\lambda} + \beta) \in \tilde{\Lambda}$. For any $(b, g) \in \Lambda$ with $g > \frac{1}{\lambda}$, $F_1(b, g) > 0$. As a result, Lemma 4.1 holds with

$$(b_0, g_0) = (\frac{\lambda-2}{2\lambda}, \frac{1}{\lambda} + \beta).$$

□

For later use, we choose $\alpha > 0$ sufficiently small such that $(1-\alpha)(b_0, g_0), (1+\alpha)(b_0, g_0) \in \tilde{\Lambda}$, $\underline{g}, \underline{g} > 0$ and

$$\begin{aligned} \lambda(1-\alpha)b_0\underline{g} &> (1+\alpha)b_0, \\ \lambda(1-\alpha)b_0[1 - (1+\alpha)(b_0 + g_0)] &> \bar{g} + \lambda(1+\alpha)b_0\bar{g}, \end{aligned} \tag{4.1}$$

where $\underline{g} = (1-\alpha)(b_0 + g_0) - (1+\alpha)b_0$ and $\bar{g} = (1+\alpha)(b_0 + g_0) - (1-\alpha)b_0$. Note that the existence of α depends on the fact that

$$\begin{aligned} \lambda b_0 g_0 &> b_0, \\ \lambda b_0 [1 - (b_0 + g_0)] &> g_0 + \lambda b_0 g_0. \end{aligned}$$

according to Lemma 4.1.

We define

$$\Lambda_2 = \{(b, g) : (1-\alpha)b_0 \leq b \leq (1+\alpha)b_0, (1-\alpha)(b_0 + g_0) \leq b + g \leq (1+\alpha)(b_0 + g_0)\},$$

then it is easy to check that

$$\underline{g} = \min\{g : (b, g) \in \Lambda_2\} \text{ and } \bar{g} = \max\{g : (b, g) \in \Lambda_2\}. \tag{4.2}$$

We define

$$\gamma = \inf\{t \geq 0 : \frac{1}{n}(B_t, G_t) \notin \Lambda_2\}$$

as the first moment that (B, G) exits $n\Lambda_2$. Furthermore, we define

$$D = \inf\{\|(b, g) - (b_0, g_0)\|_1 : (b, g) \notin \Lambda_2\} > 0.$$

The next lemma about the time (B, G) waits for to exit $n\Lambda_2$ conditioned on $(B_0, G_0) = (nb_0, ng_0)$ is utilized later.

Lemma 4.2. *For given $\lambda > 4$ and α defined as in Equation (4.1), there exists $C_4 = C_4(\lambda, \alpha) > 0$ and $N_6 = N_6(\lambda, \alpha) > 0$ such that*

$$P_{\lambda, n}^{nb_0, ng_0}\left(\gamma \geq \frac{D}{4(1+\lambda)}\right) \geq 1 - e^{-C_4 n}$$

for $n \geq N_6$.

Proof. Conditioned on $(B_0, G_0) = (nb_0, ng_0)$, the l_1 norm $\|(B, G)\|_1$ must change by at least nD for $\frac{1}{n}(B, G)$ to exit Λ_2 . At each moment that (B, G) jumps, $\|(B, G)\|_1$ changes by at most 2. Hence (B, G) must jump at least $\frac{nD}{2}$ times to exit $n\Lambda_2$. It is easy to check that (B, G) changes state with rate at most

$$n + \frac{\lambda}{n}nn = (1 + \lambda)n.$$

As a result,

$$P_{\lambda, n}^{nb_0, ng_0}\left(\gamma \leq \frac{D}{4(1 + \lambda)}\right) \leq P\left(Y_n\left(\frac{D}{4(1 + \lambda)}\right) \geq \frac{nD}{2}\right),$$

where $\{Y_n(t)\}_{t \geq 0}$ is the Poisson process with rate $(1 + \lambda)n$ for each $n \geq 1$. $Y_n(t)$ has the same probability distribution as that of $Y_1(nt)$, hence

$$P_{\lambda, n}^{nb_0, ng_0}\left(\gamma \leq \frac{D}{4(1 + \lambda)}\right) \leq P\left(\frac{Y_1\left(n\frac{D}{4(1 + \lambda)}\right)}{n} \geq \frac{D}{2}\right). \quad (4.3)$$

According to classic limit theorems of the Poisson process, for any $t > 0$, $Y_1(nt)/n$ converges to $(1 + \lambda)t$ in probability as $n \rightarrow +\infty$ and there exists $I(t) > 0$ such that

$$P\left(\frac{Y_1(nt)}{n} \geq 2(1 + \lambda)t\right) \leq e^{-nI(t)}$$

for sufficiently large n . As a result, there exists $N_6 = N_6(\lambda, \alpha)$ such that

$$P\left(\frac{Y_1\left(n\frac{D}{4(1 + \lambda)}\right)}{n} \geq \frac{D}{2}\right) \leq e^{-nI\left(\frac{D}{4(1 + \lambda)}\right)} \quad (4.4)$$

for $n \geq N_6$. Lemma 4.2 follows from Equations (4.3) and (4.4) directly with $C_4 = I\left(\frac{D}{4(1 + \lambda)}\right)$. \square

We introduce a birth-and-death process as an auxiliary model for the proof of Equation (2.1). Let $\{(\hat{B}_t, \hat{S}_t)\}_{t \geq 0}$ be the birth-and-death process with transition rates function given by

$$(\hat{B}, \hat{S}) \rightarrow \begin{cases} (\hat{B}, \hat{S}) - (1, 1) & \text{at rate } n(1 + \alpha)b_0, \\ (\hat{B}, \hat{S}) - (0, 1) & \text{at rate } n\bar{g}, \\ (\hat{B}, \hat{S}) + (1, 0) & \text{at rate } \lambda n(1 - \alpha)b_0g, \\ (\hat{B}, \hat{S}) + (0, 1) & \text{at rate } \lambda n(1 - \alpha)b_0[1 - (1 + \alpha)(b_0 + g_0)]. \end{cases}$$

Later we will show that $B_t \geq \hat{B}_t$ and $B_t + G_t \geq \hat{S}_t$ in the sense of coupling for $t \in [0, \gamma]$, for which we introduce (\hat{B}, \hat{S}) . The next lemma shows that for any $t > 0$, $\hat{S}_t \geq \hat{S}_0$ and $\hat{B}_t \geq \hat{B}_0$ with high probability.

Lemma 4.3. *There exists $C_5 = C_5(\lambda, \alpha) > 0$ such that for any $t > 0$ and $n \geq 1$,*

$$P\left(\hat{B}_t \geq nb_0, \hat{S}_t \geq ng_0 + nb_0 \mid (\hat{B}_0, \hat{S}_0) = (nb_0, ng_0 + nb_0)\right) \geq 1 - 2e^{-C_5 nt}.$$

Proof. Throughout this proof we assume that $(\hat{B}_0, \hat{S}_0) = (nb_0, nb_0 + ng_0)$. Let $q_1 = \lambda(1 - \alpha)b_0[1 - (1 + \alpha)(b_0 + g_0)]$ and $q_2 = (1 + \alpha)b_0 + \bar{g}$, then $\{\hat{S}_t\}_{t \geq 0}$ is a birth-and-death process with transition rates function given by

$$\hat{S} \rightarrow \begin{cases} \hat{S} + 1 & \text{at rate } nq_1, \\ \hat{S} - 1 & \text{at rate } nq_2. \end{cases}$$

By Equation (4.1), it is easy to check that $q_1 > q_2$, then we can choose $\frac{q_2}{q_1} < \rho < 1$. Since $\rho < 1$, by Chebyshev's inequality,

$$P(\hat{S}_t \leq nb_0 + ng_0) = P(\rho^{\hat{S}_t} \geq \rho^{nb_0 + ng_0}) \leq \rho^{-nb_0 - ng_0} E\rho^{\hat{S}_t}. \quad (4.5)$$

According to the transition rates function of \hat{S} ,

$$\begin{aligned} \frac{d}{dt} E\rho^{\hat{S}_t} &= nq_1(E\rho^{\hat{S}_t+1} - E\rho^{\hat{S}_t}) + nq_2(E\rho^{\hat{S}_t-1} - E\rho^{\hat{S}_t}) \\ &= n(q_1\rho + \frac{q_2}{\rho} - q_1 - q_2)E\rho^{\hat{S}_t}. \end{aligned}$$

Then,

$$E\rho^{\hat{S}_t} = \rho^{nb_0 + ng_0} e^{-C_6 nt} \quad (4.6)$$

since $\hat{S}_0 = nb_0 + ng_0$, where $C_6 = q_1 + q_2 - q_1\rho - \frac{q_2}{\rho}$. Note that $C_6 > 0$ since $\frac{q_2}{q_1} < \rho < 1$. By Equations (4.5) and (4.6),

$$P(\hat{S}_t \leq nb_0 + ng_0) \leq e^{-C_6 nt}. \quad (4.7)$$

According to the same analysis as that gives Equation (4.7), there exists $C_7 > 0$ such that

$$P(\hat{B}_t \leq nb_0) \leq e^{-C_7 nt}. \quad (4.8)$$

Let $C_5 = \min\{C_6, C_7\}$, then Lemma 4.3 follows from Equations (4.7) and (4.8) directly. \square

The next lemma shows that B_t and $B_t + G_t$ are bounded from below by \hat{B}_t and \hat{S}_t respectively for $t \geq [0, \gamma]$.

Lemma 4.4. *Conditioned on $(B_0, B_0 + G_0) = (\hat{B}_0, \hat{S}_0) = (nb_0, nb_0 + ng_0)$,*

$$B_t \geq \hat{B}_t \text{ and } B_t + G_t \geq \hat{S}_t$$

in the sense of coupling for $t \in [0, \gamma]$.

Proof. For $t \in [0, \gamma]$, $\frac{1}{n}(B_t, G_t) \in \Lambda_2$. The transition rates function of $\{(B_t, B_t + G_t)\}_{t \geq 0}$ is given by

$$(B, B + G) \rightarrow \begin{cases} (B, B + G) - (1, 1) & \text{at rate } F_1(B, G) = B, \\ (B, B + G) - (0, 1) & \text{at rate } F_2(B, G) = G, \\ (B, B + G) + (1, 0) & \text{at rate } F_3(B, G) = \frac{\lambda}{n}BG, \\ (B, B + G) + (0, 1) & \text{at rate } F_4(B, G) = \frac{\lambda}{n}B(n - B - G). \end{cases}$$

According to the definition of Λ_2 , for any (B, G) that $\frac{1}{n}(B, G) \in \Lambda_2$,

$$\begin{aligned} F_1(B, G) &\leq n(1 + \alpha)b_0, \quad F_2(B, G) \leq n\bar{g}, \\ F_3(B, G) &\geq \lambda n(1 - \alpha)b_0\underline{g}, \quad F_4(B, G) \geq \lambda n(1 - \alpha)b_0[1 - (1 + \alpha)(b_0 + g_0)]. \end{aligned}$$

As a result, we can couple $(B_t, B_t + G_t)$ and $(\widehat{B}_t, \widehat{S}_t)$ as follows for $t \in [0, \gamma]$.

$$(B, B + G, \widehat{B}, \widehat{S}) \rightarrow \begin{cases} (B, B + G, \widehat{B}, \widehat{S}) - (1, 1, 1, 1) & \text{at rate } F_1(B, G), \\ (B, B + G, \widehat{B}, \widehat{S}) - (0, 0, 1, 1) & \text{at rate } n(1 + \alpha)b_0 - F_1(B, G), \\ (B, B + G, \widehat{B}, \widehat{S}) - (0, 1, 0, 1) & \text{at rate } F_2(B, G), \\ (B, B + G, \widehat{B}, \widehat{S}) - (0, 0, 0, 1) & \text{at rate } n\bar{g} - F_2(B, G), \\ (B, B + G, \widehat{B}, \widehat{S}) + (1, 0, 1, 0) & \text{at rate } \lambda n(1 - \alpha)b_0\bar{g}, \\ (B, B + G, \widehat{B}, \widehat{S}) + (1, 0, 0, 0) & \text{at rate } F_3(B, G) - \lambda n(1 - \alpha)b_0\bar{g}, \\ (B, B + G, \widehat{B}, \widehat{S}) + (0, 1, 0, 1) & \text{at rate } \lambda n(1 - \alpha)b_0[1 - (1 + \alpha)(b_0 + g_0)], \\ (B, B + G, \widehat{B}, \widehat{S}) + (0, 1, 0, 0) & \text{at rate } F_4(B, G) - \lambda n(1 - \alpha)b_0[1 - (1 + \alpha)(b_0 + g_0)]. \end{cases}$$

The above coupling does not change the property that $B \geq \widehat{B}$ and $B + G \geq \widehat{S}$, hence the proof is complete. \square

At last we give the proof of Equation (2.1).

Proof of Equation (2.1). We define

$$\Lambda_1 = \{(b, g) \in \Lambda : b \geq b_0, b + g \geq b_0 + g_0\}$$

as we have done at the beginning of this section.

Let $T_4 = \frac{D}{4(1+\lambda)} > 0$, where D is defined as in Lemma 4.2, then the first step of this proof is to show that

$$P_{\lambda, n}^{nb_0, ng_0} \left(\frac{(B_{T_4}, G_{T_4})}{n} \in \Lambda_1 \right) \geq 1 - 3e^{-C_8 n} \quad (4.9)$$

for some $C_8 = C_8(\lambda) > 0$ and sufficiently large n . The proof of Equation (4.9) is as follows. By Lemma 4.4, $B_t \geq \widehat{B}_t$ and $B_t + G_t \geq \widehat{S}_t$ for $t \in [0, \gamma]$, where $(B_0, G_0 + B_0) = (\widehat{B}_0, \widehat{S}_0) = (nb_0, nb_0 + ng_0)$. Therefore,

$$\begin{aligned} P_{\lambda, n}^{nb_0, ng_0} \left(\frac{(B_{T_4}, G_{T_4})}{n} \in \Lambda_1 \right) &\geq P_{\lambda, n}^{nb_0, ng_0} \left(\frac{(B_{T_4}, G_{T_4})}{n} \in \Lambda_1, \gamma \geq T_4 \right) \\ &= P_{\lambda, n}^{nb_0, ng_0} (B_{T_4} \geq nb_0, B_{T_4} + G_{T_4} \geq nb_0 + ng_0, \gamma \geq T_4) \\ &\geq P_{\lambda, n}^{nb_0, ng_0} (\widehat{B}_{T_4} \geq nb_0, \widehat{S}_{T_4} \geq nb_0 + ng_0, \gamma \geq T_4) \\ &\geq P(\widehat{B}_{T_4} \geq nb_0, \widehat{S}_{T_4} \geq nb_0 + ng_0) - P_{\lambda, n}^{nb_0, ng_0}(\gamma < T_4) \\ &\geq 1 - 2e^{-C_5 T_4 n} - e^{-C_4 n} \end{aligned} \quad (4.10)$$

for $n \geq N_6$, where N_6, C_4 are defined as in Lemma 4.2 while C_5 is defined as in Lemma 4.3. Note that the last inequality in Equation (4.10) follows from Lemmas 4.2 and 4.3. As a result, Equation (4.9) holds with $C_8 = \min\{C_5 T_4, C_4\}$ and $n \geq N_6$.

Note that if $(B_1, G_1) \succeq (B_2, G_2)$ and $\frac{1}{n}(B_2, G_2) \in \Lambda_1$, then $\frac{1}{n}(B_1, G_1) \in \Lambda_1$. Therefore, by Lemma 2.2,

$$P_{\lambda, n}^{nb_0, ng_0} \left(\frac{(B_{T_4}, G_{T_4})}{n} \in \Lambda_1 \right) \leq P_{\lambda, n}^{B, G} \left(\frac{(B_{T_4}, G_{T_4})}{n} \in \Lambda_1 \right)$$

for any $(B, G) \succeq (nb_0, ng_0)$. Then by Equation (4.9),

$$P_{\lambda, n}^{B, G}((B_{T_4}, G_{T_4}) \succeq (nb_0, ng_0)) = P_{\lambda, n}^{B, G}\left(\frac{(B_{T_4}, G_{T_4})}{n} \in \Lambda_1\right) \geq 1 - 3e^{-C_8 n} \quad (4.11)$$

for any $(B, G) \succeq (nb_0, ng_0)$ and $n \geq N_6$. By Equation (4.11), utilizing the Markov property for $e^{\frac{C_8}{2}n}$ times,

$$P_{\lambda, n}^{nb_0, ng_0}\left((B_{T_4 e^{\frac{C_8}{2}n}}, G_{T_4 e^{\frac{C_8}{2}n}}) \succeq (nb_0, ng_0)\right) \geq (1 - 3e^{-C_8 n})^{e^{\frac{C_8}{2}n}} \geq 1 - 3e^{-\frac{C_8}{2}n}$$

for $n \geq N_6$ and hence

$$P_{\lambda, n}^{nb_0, ng_0}(\tau > T_4 e^{\frac{C_8}{2}n}) \geq 1 - 3e^{-\frac{C_8}{2}n} \quad (4.12)$$

for $n \geq N_6$. By Equation (4.12),

$$P_{\lambda, n}^{nb_0, ng_0}(\tau > e^{\frac{C_8}{4}n}) \geq 1 - 3e^{-\frac{C_8}{2}n} \quad (4.13)$$

for sufficiently large n . Then, by Lemma 2.2 and the fact that $(n, 0) \succeq (nb_0, ng_0)$,

$$P_{\lambda, n}^{n, 0}(\tau > e^{\frac{C_8}{4}n}) \geq P_{\lambda, n}^{nb_0, ng_0}(\tau > e^{\frac{C_8}{4}n}) \geq 1 - 3e^{-\frac{C_8}{2}n}$$

for sufficiently large n and hence Equation (2.1) holds with $C(\lambda) = \frac{C_8}{4}$. □

5 Proof of Equation (2.2)

In this section we give the proof of Equation (2.2). It is obviously that we only need to deal with small θ , so we assume that $\theta < 1$. Throughout this section we use $\{(b_t, g_t)\}_{t \geq 0}$ to denote the solution to ODE (3.1) with initial condition $(b_0, g_0) = (1, 0)$. Sometimes we write b_t, g_t as $b(t), g(t)$ when the subscript is complex. First we show that $\frac{1}{n}\|(B_t, G_t)\|_1$ stays small for $O(\sqrt{n})$ units of time with high probability when $\lambda < 4$.

Lemma 5.1. *For given $\lambda < 4$ and $\theta \in (0, 1)$, there exist $T_1 = T_1(\lambda, \theta)$, $N_2 = N_2(\lambda, \theta)$, $C_2 = C_2(\lambda, \theta)$ and $C_3 = C_3(\lambda, \theta)$ such that*

$$P_{\lambda, n}^{n, 0}\left(\sup_{0 \leq t \leq C_2 \sqrt{n}} \frac{\|(B_{T_1+t}, G_{T_1+t})\|_1}{n} \geq \frac{\theta}{(3+\theta)\lambda}\right) \leq \frac{C_3}{\sqrt{n}}$$

for each $n \geq N_2$.

Proof. By Lemma 3.3, for $\lambda < 4$, we can choose $0 < T_3(\lambda, \theta) < T_1(\lambda, \theta)$ such that

$$b(T_3) + g(T_3) \leq \frac{\theta}{2(3+\theta)\lambda}, \quad b(T_1) + g(T_1) \leq \frac{b(T_3)}{2}$$

and

$$b_t + g_t \leq \frac{\theta}{2(3+\theta)\lambda}$$

for any $t \in [T_3, T_1]$. Let $\{(\widehat{b}_t, \widehat{g}_t)\}_{t \geq 0}$ be the solution to ODE (3.1) with $(\widehat{b}_0, \widehat{g}_0) = (b(T_3), g(T_3))$, then $(\widehat{b}_t, \widehat{g}_t) = (b(T_3 + t), g(T_3 + t))$ and hence

$$\widehat{b}_t + \widehat{g}_t \leq \frac{\theta}{2(3 + \theta)\lambda} \quad (5.1)$$

for $t \in [0, T_1 - T_3]$. Since $\frac{d}{dt}b_t \geq -b_t$, $b_t \geq b_0 e^{-t}$ and hence $b(T_3) > 0$. Let $T = T_1 - T_3$ and $\epsilon_1 = \min\{\frac{\theta}{5(3+\theta)\lambda}, \frac{b(T_3)}{10}\} > 0$, then according to Lemma 3.2,

$$P_{\lambda,n}^{nb(T_3), ng(T_3)} \left(\sup_{0 \leq t \leq T} \left\| \frac{(B_t, G_t)}{n} - (\widehat{b}_t, \widehat{g}_t) \right\|_1 > \epsilon_1 \right) \leq \frac{C_1(T_1 - T_3, \epsilon_1)}{n} \quad (5.2)$$

for $n \geq N_1(T_1 - T_3, \epsilon_1)$, where C_1 and N_1 are as defined in Lemma 3.2. By Equations (5.1) and (5.2),

$$P_{\lambda,n}^{nb(T_3), ng(T_3)} \left(\sup_{0 \leq t \leq T_1 - T_3} \frac{\|(B_t, G_t)\|_1}{n} \leq \frac{7\theta}{10(3 + \theta)\lambda} \right) \quad (5.3)$$

$$\text{and } \left(nb(T_3), ng(T_3) \right) \succeq \left(B(T_1 - T_3), G(T_1 - T_3) \right) \geq 1 - \frac{C_1(T_1 - T_3, \epsilon_1)}{n}$$

for $n \geq N_1(T_1 - T_3, \epsilon_1)$. By Lemma 2.2 and Equation (5.3), for any $(B, G) \preceq (nb(T_3), ng(T_3))$,

$$P_{\lambda,n}^{B,G} \left(\sup_{0 \leq t \leq T_1 - T_3} \frac{\|(B_t, G_t)\|_1}{n} \leq \frac{7\theta}{10(3 + \theta)\lambda} \right) \quad (5.4)$$

$$\text{and } \left(nb(T_3), ng(T_3) \right) \succeq \left(B(T_1 - T_3), G(T_1 - T_3) \right) \geq 1 - \frac{C_1(T_1 - T_3, \epsilon_1)}{n}$$

for $n \geq N_1(T_1 - T_3, \epsilon_1)$. By Equation (5.4) and utilizing the Markov property for \sqrt{n} times,

$$\begin{aligned} P_{\lambda,n}^{nb(T_3), ng(T_3)} \left(\sup_{0 \leq t \leq (T_1 - T_3)\sqrt{N}} \frac{\|(B_t, G_t)\|_1}{n} \leq \frac{7\theta}{10(\theta + 3)\lambda} \right) &\geq \left(1 - \frac{C_1(T_1 - T_3, \epsilon_1)}{n} \right)^{\sqrt{n}} \\ &\geq 1 - \frac{C_1(T_1 - T_3, \epsilon_1)}{\sqrt{n}} \end{aligned} \quad (5.5)$$

for $n \geq N_1(T_1 - T_3, \epsilon_1)$. By Lemma 3.2,

$$P_{\lambda,n}^{n,0} \left(\left\| \frac{(B(T_1), G(T_1))}{n} - (b(T_1), g(T_1)) \right\|_1 \leq \frac{b(T_3)}{10} \right) \geq 1 - \frac{C_1(T_1, \frac{b(T_3)}{10})}{n}$$

for $n \geq N_1(T_1, \frac{b(T_3)}{10})$ and hence

$$P_{\lambda,n}^{n,0} \left((B(T_1), G(T_1)) \preceq (nb(T_3), ng(T_3)) \right) \geq 1 - \frac{C_1(T_1, \frac{b(T_3)}{10})}{n} \quad (5.6)$$

for $n \geq N_1(T_1, \frac{b(T_3)}{10})$. Conditioned on $(B(T_1), G(T_1)) \preceq (nb(T_3), ng(T_3))$, according to Lemma 2.2 and Equation (5.5),

$$\sup_{0 \leq t \leq (T_1 - T_3)\sqrt{N}} \frac{\|(B_{T_1+t}, G_{T_1+t})\|_1}{n} \leq \frac{7\theta}{10(3 + \theta)\lambda}$$

with probability at least $1 - \frac{C_1(T_1 - T_3, \epsilon_1)}{\sqrt{n}}$ for $n \geq N_1(T_1 - T_3, \epsilon_1)$. Hence by Equation (5.6),

$$\sup_{0 \leq t \leq (T_1 - T_3)\sqrt{N}} \frac{\|(B_{T_1+t}, G_{T_1+t})\|_1}{n} \leq \frac{7\theta}{10(3+\theta)\lambda} < \frac{\theta}{(3+\theta)\lambda}$$

with probability at least $1 - \frac{C_1(T_1 - T_3, \epsilon_1)}{\sqrt{n}} - \frac{C_1(T_1, \frac{b(T_3)}{10})}{n}$ for $n \geq \max\{N_1(T_1, \frac{b(T_3)}{10}), N_1(T_1 - T_3, \epsilon_1)\}$. We choose N_3 such that

$$\frac{C_1(T_1, \frac{b(T_3)}{10})}{n} \leq \frac{1}{\sqrt{n}}$$

for $n \geq N_3$, then Lemma 5.1 holds with $T_1 = T_1(\lambda, \theta)$, $N_2 = \max\{N_3, N_1(T_1, \frac{b(T_3)}{10}), N_1(T_1 - T_3, \epsilon_1)\}$, $C_2 = T_1 - T_3$ and $C_3 = 1 + C_1(T_1 - T_3, \epsilon_1)$. \square

To prove Equation (2.2), we introduce the following Markov process as an auxiliary model.

Let $\{\tilde{B}_t\}_{t \geq 0}$ be a continuous-time Markov process with state space $\{0, 1, 2, \dots\}$ and transition rates function given by

$$\tilde{B} \rightarrow \begin{cases} \tilde{B} - 1 & \text{at rate } \tilde{B}, \\ \tilde{B} + 1 & \text{at rate } \frac{\theta \tilde{B}}{\theta + 3}, \end{cases}$$

then the following lemma shows that \tilde{B} dies out in $O(\log n)$ units of time with high probability conditioned on $\tilde{B}_0 = n$.

Lemma 5.2.

$$P(\tilde{B}_{(1+\frac{\theta}{2})\log n} = 0 | \tilde{B}_0 = n) \geq 1 - n^{-\frac{\theta}{2\theta+6}}$$

for each $n \geq 1$.

Proof. Let $h(t) = E(\tilde{B}_t | \tilde{B}_0 = n)$, then according to the transition rates function of \tilde{B} ,

$$\frac{d}{dt}h(t) = -h(t) + \frac{\theta h(t)}{\theta + 3} = -\frac{3h(t)}{\theta + 3}.$$

Then, $h(t) = ne^{-\frac{3t}{\theta+3}}$ since $h(0) = n$. By Chebyshev's inequality,

$$P(\tilde{B}_{(1+\frac{\theta}{2})\log n} \geq 1 | \tilde{B}_0 = n) \leq h((1+\frac{\theta}{2})\log n) = n^{-\frac{\theta}{2\theta+6}}.$$

Therefore,

$$P(\tilde{B}_{(1+\frac{\theta}{2})\log n} = 0 | \tilde{B}_0 = n) = 1 - P(\tilde{B}_{(1+\frac{\theta}{2})\log n} \geq 1 | \tilde{B}_0 = n) \geq 1 - n^{-\frac{\theta}{2\theta+6}}.$$

\square

At last we give the proof of Equation (2.2). From now on we assume that $\tilde{B}_0 = n$.

Proof of Equation (2.2). Let T_1 be as defined in Lemma 5.1. On the event $\|(B_{T_1}, G_{T_1})\|_1 < \frac{n\theta}{(\theta+3)\lambda}$, we define

$$\sigma = \inf\{t \geq 0 : \|(B_{T_1+t}, G_{T_1+t})\|_1 \geq \frac{n\theta}{(\theta+3)\lambda}\}.$$

For $t \in [T_1, T_1 + \sigma]$, $B \rightarrow B + 1$ at rate

$$\frac{\lambda}{n}BG \leq \frac{\lambda}{n}B \frac{n\theta}{(\theta+3)\lambda} = \frac{\theta B}{\theta+3}.$$

Therefore, conditioned on $\|(B_{T_1}, G_{T_1})\|_1 < \frac{n\theta}{(\theta+3)\lambda}$,

$$B_{T_1+t} \leq \tilde{B}_t$$

for $t \in [0, \sigma]$ in the sense of coupling. As a result,

$$\begin{aligned} & P_{\lambda,n}^{n,0} \left(B_{T_1+(1+\frac{\theta}{2})\log n} = 0, \|(B_{T_1}, G_{T_1})\|_1 < \frac{n\theta}{(\theta+3)\lambda}, \sigma > (1+\frac{\theta}{2})\log n \right) \\ & \geq P_{\lambda,n}^{n,0} \left(\tilde{B}_{(1+\frac{\theta}{2})\log n} = 0, \|(B_{T_1}, G_{T_1})\|_1 < \frac{n\theta}{(\theta+3)\lambda}, \sigma > (1+\frac{\theta}{2})\log n \right) \\ & \geq P(\tilde{B}_{(1+\frac{\theta}{2})\log n} = 0) - P_{\lambda,n}^{n,0} \left(\sup_{0 \leq t \leq (1+\frac{\theta}{2})\log n} \|(B_{T_1+t}, G_{T_1+t})\|_1 \geq \frac{n\theta}{(\theta+3)\lambda} \right). \end{aligned} \quad (5.7)$$

We choose N_4 such that $(1+\frac{\theta}{2})\log n \leq C_2\sqrt{n}$ for each $n \geq N_4$, then by Lemma 5.1,

$$P_{\lambda,n}^{n,0} \left(\sup_{0 \leq t \leq (1+\frac{\theta}{2})\log n} \|(B_{T_1+t}, G_{T_1+t})\|_1 \geq \frac{n\theta}{(\theta+3)\lambda} \right) \leq \frac{C_3}{\sqrt{n}} \quad (5.8)$$

for $n \geq N_5 = \max\{N_4, N_2\}$. By Equations (5.7), (5.8) and Lemma 5.2,

$$P_{\lambda,n}^{n,0} \left(B_{T_1+(1+\frac{\theta}{2})\log n} = 0 \right) \geq 1 - \frac{C_3}{\sqrt{n}} - n^{-\frac{\theta}{2\theta+6}} \quad (5.9)$$

for $n \geq N_5$. By Equation (5.9),

$$P_{\lambda,n}^{n,0} \left(\tau \leq T_1 + (1+\frac{\theta}{2})\log n \right) \geq 1 - \frac{C_3}{\sqrt{n}} - n^{-\frac{\theta}{2\theta+6}} \quad (5.10)$$

for $n \geq N_5$. Equation (2.2) follows from Equation (5.10) directly since

$$T_1 + (1+\frac{\theta}{2})\log n \leq (1+\theta)\log n$$

for sufficiently large n . □

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